Sec. 11.4 Rational Functions

Rational function – has the form $R(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomial functions and q is not the zero polynomial. The domain consists of all real numbers except for those in which the denominator q is 0.

Lowest Terms – when p and q have no common factors, the zeros of the numerator are the x— intercepts of the graph of R.

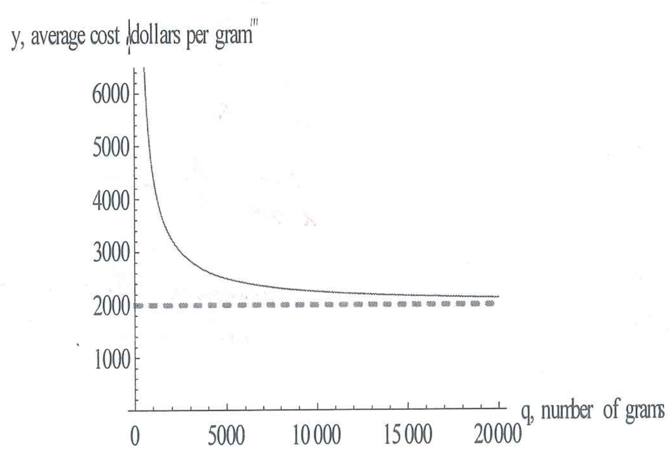
Unbounded in the positive (or negative) direction – as x approaches 0 the values of R(x) become larger and larger (or smaller and smaller) but never actually stop

- Symbolized by $R(\mathrm{x}) \to \infty$ read as $R(x)$ approaches infinity
- In calculus, these are called limits and symbolized as $\lim\limits_{x\to\infty} R(x) = \infty$ and read as the limit of $R(x)$ as x approaches 0 is infinity which means that $R(x) \to \infty$ as $x \to 0$

Ex: A pharmaceutical company wants to begin production of a new drug. The total cost C, in dollars, of making q grams of the drug is given by the linear function

$$C(q) = 2{,}500{,}000 + 2000q.$$

The fact that $C(0) = 2{,}500{,}000$ tells us that the company spends \$2,500,000 before it starts making the drug. This quantity is known as the *fixed cost* because it does not depend on how much of the drug is made. It represents the cost for research, testing, and equipment. In addition, the slope of C tells us that each gram of the drug costs an extra \$2000 to make. This quantity is known as the *variable cost* per unit. It represents the additional cost, in labor and materials, to make an additional gram of the drug. We define the average cost, $a(q)$, as the cost per gram to produce q grams ($q > 0$) of the drug: $a(q) = \dfrac{\text{Total cost}}{\text{Number of grams}} = \dfrac{C(q)}{q} = \dfrac{2{,}500{,}000 + 2000q}{q}$.



Ex: What is the domain of each?

a. $R(x) = \dfrac{2x^2 - 4}{x^2 - 5}$

$x^2 - 5 \neq 0$

$x^2 \neq 5$

$\boxed{x \neq \pm\sqrt{5}}$

b. $f(x) = \dfrac{2x^2 - 4}{x^2 + 3}$

$x^2 + 3 \neq 0$

$x^2 \neq -3$

IMPOSSIBLE

$\boxed{\text{ALL REAL NUMBERS}}$

c. $g(x) = \dfrac{x^2 - 4}{x - 2}$ $\qquad \dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$ $\quad x+2$

$x - 2 \neq 0$

$\boxed{x \neq 2}$

USE ORIGINAL DENOMINATOR!

For x of large enough magnitude (either positive or negative), the graph of the rational function r looks like the graph of a power function. If $r(x) = \dfrac{p(x)}{q(x)}$, then the **long-run behavior** of $y = r(x)$ is given by: $\quad y = \dfrac{\text{Leading term of } p}{\text{Leading term of } q}$

Using limits, we write: $\quad \lim_{x\to\pm\infty} \dfrac{p(x)}{q(x)} = \lim_{x\to\pm\infty} \dfrac{\text{Leading term of p}}{\text{Leading term of q}}$

This limit, if it exists, gives the horizontal asymptote of $r(x)$.

Ex: Describe the long-run behavior (write using limit notation) of:

a.) $f(x) = \dfrac{6x^4 + x^3 + 1}{-5x + 2x^2}$

$= \dfrac{6x^4 + 3x + 1}{2x^2 - 5x}$

$y = \dfrac{6x^4}{2x^2}$

$y = 3x^2$

$\lim_{x\to\infty} 3x^2 = \infty$

$\lim_{x\to-\infty} 3x^2 = \infty$

No horizontal asymptote

b) $r(x) = \dfrac{x+3}{x+2}$ for x > 0

$y = \dfrac{x}{x}$

$y = 1$

$\lim_{x\to\infty} 1 = 1$

so

$\lim_{x\to\infty} \dfrac{x+3}{x+2} = 1$

Horizontal Asymptote at $y = 1$

c.) $g(x) = \dfrac{3x+1}{x^2 + x - 2}$ for $x > 0$

$y = \dfrac{3x}{x^2}$

$y = \dfrac{3}{x}$

$\lim_{x\to\infty} \dfrac{3}{x} = 0$

Horizontal Asymptote at $y = 0$

Finding Asymptotes:

1. If as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then line $y = L$ is a horizontal asymptote.
2. To find horizontal asymptotes, identify the degree of the numerator and the degree of the denominator.
   a. If the function is proper (degree of numerator is smaller than degree of denominator), then $y = 0$ is a horizontal asymptote.
   b. If the function has the degree of the numerator equal to the degree of the denominator, then $y = \frac{a_n}{b_m}$ where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator (HORIZONTAL ASYMPTOTE).
   c. If the degree of the numerator is one greater than the degree of the denominator, you need to use long division to find the quotient and it will be in the form $y = ax + b$ (OBLIQUE ASYMPTOTE).
   d. If the degree of the numerator is more than one greater than the degree of the denominator, you will have no horizontal or oblique asymptotes.
3. If as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a vertical asymptote of the graph of R.
4. To find vertical asymptotes, make sure the rational function $R(x)$ is in lowest terms. Then find the values for which the denominator = 0. These values will be your vertical asymptotes.

Ex: Find the vertical, horizontal and/or oblique asymptotes for the following functions. Then state their domains.

a. $R(x) = \dfrac{x^2}{x^2+1}$

HA: $y = \frac{1}{1}$ $\boxed{y = 1}$

$x^2 + 1 = 0$

$x^2 = -1$

NOT POSSIBLE $\boxed{\text{NO VA.}}$

b. $G(x) = \dfrac{x}{x^2-4}$

$\dfrac{x}{(x+2)(x-2)}$

VA: $\boxed{x = -2 \quad x = 2}$

$y = \dfrac{x}{x^2}$

$y = \dfrac{1}{x}$

$\lim_{x \to \pm\infty} \frac{1}{x} = 0$

HA: $\boxed{y = 0}$

c. $F(x) = \dfrac{3x^4+4}{x^3+3x}$

$\dfrac{3x^4+4}{x(x^2+3)}$

$y = \dfrac{3x^4}{x^3}$

$\boxed{\text{HA: } x = 0}$

$x^2 + 3 = 0$

$x^2 = -3$

Not Possible

$\boxed{\text{OA: } y = 3x}$

d. $B(x) = \dfrac{x^3}{x-1}$

$y = \dfrac{x^3}{x} = x^2$

$\boxed{\text{NO HA ASYMPTOTES}}$

$\lim_{x \to \infty} = \infty$ $\quad \lim_{x \to -\infty} = \infty$

VA: $x - 1 \neq 0$

$\boxed{x = 1}$

HW: pg 458-461 #S1-S4,S9,3-6,8-13,15-18,22*,25
*Read through the book's explanation for part c.